

AN IMPROVED JULIA-CARATHEODORY THEOREM FOR SCHUR-AGLER MAPPINGS OF THE UNIT BALL

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ABSTRACT. We adapt Sarason's proof of the Julia-Caratheodory theorem to the class of Schur-Agler mappings of the unit ball, obtaining a strengthened form of this theorem. In particular those quantities which appear in the classical theorem and depend only on the component of the mapping in the complex normal direction have K -limits (not just restricted K -limits) at the boundary.

Let \mathbb{B}^n denote the open unit ball in n -dimensional complex space. In this note we show that holomorphic mappings $\varphi : \mathbb{B}^n \rightarrow \mathbb{B}^m$ belonging to the *Schur-Agler class* (defined below) satisfy a strengthened form of the Julia-Caratheodory theorem (Theorem 1.9). While the Schur-Agler class has received much attention in the past several years from operator theorists, relatively little seems to be known about the function-theoretic behavior of this class.

For many operator theoretic applications, the Schur-Agler classes $\mathcal{S}(n, 1)$ and $\mathcal{S}(n, n)$ are more appropriate analogues of the unit ball of $H^\infty(\mathbb{D})$ than are the larger classes $\text{Hol}(\mathbb{B}^n, \mathbb{D})$ and $\text{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. For example, the Schur-Agler class is a natural setting for multivariable versions of von Neumann's inequality [5], the Sz.-Nagy dilation theorem [3], commutant lifting theorems [4] and the Nevanlinna-Pick interpolation theorem [1]. Additionally, every self-map of the ball belonging to the Schur-Agler class induces a bounded composition operator on the standard holomorphic function spaces [6], which is not true of general self-maps of the ball. This last fact suggests that mappings in the Schur-Agler class should also enjoy function-theoretic privileges over generic maps of the ball, and is the motivation for this paper.

Indeed there seems to be little known about the function theory of $\mathcal{S}(n, m)$ apart from what is true generically. Recently Anderson, Dritschel and Rovnyak [2] have established a family of inequalities for derivatives of Schur-Agler functions, though it is not known if these inequalities hold generically. In this paper we show that the Schur-Agler class satisfies a form of the Julia-Caratheodory theorem that is strictly

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stronger than what is true for general holomorphic functions on the unit ball. The result is proved by adapting Sarason's Hilbert space proof of the classical Julia-Caratheodory theorem [9, Chapter VI] to the ball. In fact Sarason's proof cannot prove the general Julia-Caratheodory theorem in higher dimensions, since it exploits the positivity of the de Branges-Rovnyak kernel. The analogous kernel in several variables need no longer be positive, but since the Schur-Agler class is precisely the class for which this kernel is positive, the proof goes through but in fact proves a stronger result.

Definition 1.1. *The Schur-Agler class $\mathcal{S}(n, m)$ is the set of all holomorphic mappings $\varphi : \mathbb{B}^n \rightarrow \mathbb{B}^m$ such that the Hermitian kernel*

$$k^\varphi(z, w) = \frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle}$$

is positive semidefinite.

The kernel k^φ is called the *de Branges-Rovnyak kernel* associated to φ . We let H_n^2 denote the Hilbert space of holomorphic functions on \mathbb{B}^n with reproducing kernel

$$k(z, w) = \frac{1}{1 - \langle z, w \rangle}$$

When $n > 1$ the space H_n^2 is strictly smaller than the classical Hardy space H^2 (defined by spherical means); however in many ways it is the higher-variable analogue of $H^2(\mathbb{D})$ appropriate for multivariable operator theory, see e.g. [1, 3, 4]. In this context, as mentioned above, the Schur-Agler classes play the role of the unit ball of the algebra of bounded analytic functions in \mathbb{D} , though we stress that when $n > 1$ the inclusion $\mathcal{S}(n, m) \subset \text{Hol}(\mathbb{B}^n, \mathbb{B}^m)$ is always proper.

Given a Schur-Agler mapping $\varphi \in \mathcal{S}(n, m)$, we can define another Hilbert function space $\mathcal{H}(\varphi)$ to be the space of holomorphic functions on \mathbb{B}^n with reproducing kernel k^φ . This space is always contractively contained in H_n^2 :

Lemma 1.2. *If $\varphi \in \mathcal{S}(n, m)$ and $f \in \mathcal{H}(\varphi)$ then $f \in H_n^2$ and*

$$\|f\|_{H_n^2} \leq \|f\|_{\mathcal{H}(\varphi)}$$

Proof. The positivity of k^φ implies that the operator

$$T : (f_1, \dots, f_m) \rightarrow \sum_{k=1}^m \varphi_k f_k$$

is contractive from the direct sum of m copies of H_n^2 to H_n^2 . The de Branges-Rovnyak kernel may then be written as

$$k^\varphi(z, w) = \langle (I - TT^*)^{1/2} k_w, (I - TT^*)^{1/2} k_z \rangle_{H_n^2}$$

Now let $f \in \mathcal{H}(\varphi)$. It follows from the standard de Branges-Rovnyak construction applied to T [9, Chapter 1] that there exists $g \in H_n^2$ such that $f = (I - TT^*)^{1/2} g$ and $\|f\|_{\mathcal{H}(\varphi)} = \|g\|_{H_n^2}$. Thus $f \in H_n^2$ and $\|f\|_{\mathcal{H}(\varphi)} \geq \|f\|_{H_n^2}$. \square

We will be examining the boundary behavior of Schur-Agler mappings and to a lesser extent the behavior of functions in $\mathcal{H}(\varphi)$. We recall here some basic notions in the study of boundary behavior of holomorphic functions on the unit ball, and refer to Rudin [8, Chapter 8] (or Krantz [7, Section 8.6]) for details.

Given a point $\zeta \in \partial \mathbb{B}^n$ and a real number $\alpha > 0$, the *Koranyi region* $D_\alpha(\zeta)$ is the set

$$D_\alpha(\zeta) = \{z \in \mathbb{B}^n : |1 - \langle z, \zeta \rangle| \leq \frac{\alpha}{2}(1 - |z|^2)\}$$

A function $f : \mathbb{B}^n \rightarrow \mathbb{C}$ has *K-limit* L at ζ if $\lim_{z \rightarrow \zeta} f(z) = L$ whenever z tends to ζ within a Koranyi region. Note that when $n = 1$, a *K-limit* is just a nontangential limit; however for $n > 1$ *K-limits* allow for parabolic approach in directions orthogonal to ζ . We shall also require the notion of a *restricted K-limit*: to define this, fix a point $\zeta \in \partial \mathbb{B}^n$ and consider a curve $\Gamma : [0, 1) \rightarrow \mathbb{B}^n$ such that $\Gamma(t) \rightarrow \zeta$ as $t \rightarrow 1$. Let $\gamma(t) = \langle \Gamma(t), \zeta \rangle \zeta$ be the projection of Γ onto the complex line through ζ . The curve Γ is called *special* if

$$(1) \quad \lim_{t \rightarrow 1} \frac{|\Gamma - \gamma|^2}{1 - |\gamma|^2} = 0$$

and *restricted* if it is special and in addition

$$(2) \quad \frac{|\zeta - \gamma|}{1 - |\gamma|^2} \leq A$$

for some constant $A > 0$. We say that a function $f : \mathbb{B}^n \rightarrow \mathbb{C}$ has *restricted K-limit* L at ζ if $\lim_{z \rightarrow \zeta} f(z) = L$ along every restricted curve.

Lemma 1.3. *If $f \in H_n^2$ then*

$$|f(z)| = o((1 - |z|^2)^{-1/2})$$

as $|z| \rightarrow 1$.

Proof. The Hilbert space norm of the reproducing kernel k_z is

$$\sqrt{k(z, z)} = (1 - |z|^2)^{-1/2}.$$

The statement of the lemma is thus equivalent to the statement that the normalized kernel functions $\tilde{k}_z = k_z/\|k_z\|$ tend weakly to 0 as $|z| \rightarrow 1$. That this is the case follows readily from two observations: 1) if $f \in H_n^2$ is bounded, then $\langle f, \tilde{k}_z \rangle \rightarrow 0$ since $\|k_z\| \rightarrow \infty$, and 2) the bounded functions belonging to H_n^2 (e.g. the polynomials) are norm dense in H_n^2 . \square

Proposition 1.4. *Suppose $\varphi \in \mathcal{S}(n, m)$ and $\zeta \in \partial\mathbb{B}^n$. If*

$$h(z) = \frac{1 - \langle \varphi(z), \xi \rangle}{1 - \langle z, \zeta \rangle}$$

belongs to $\mathcal{H}(\varphi)$ for some $\xi \in \mathbb{C}^m$, then $|\xi| = 1$ and φ has K -limit ξ at ζ .

Proof. If $h \in \mathcal{H}(\varphi)$ then by growth lemma $|h(z)| = o((1 - |z|^2)^{-1/2})$. So

$$|1 - \langle \varphi(z), \xi \rangle| = o\left(\frac{|1 - \langle z, \zeta \rangle|}{1 - |z|^2}(1 - |z|^2)^{1/2}\right)$$

which goes to 0 as $z \rightarrow \zeta$ within a Koranyi region; this establishes the claim. \square

We are interested in Schur-Agler mappings satisfying the following condition, which we call condition (C) following Sarason:

$$(C) \quad L = \liminf_{z \rightarrow \zeta} \frac{1 - |\varphi(z)|^2}{1 - |z|^2} < \infty$$

The following is then the analogue, for the Schur-Agler class, of Sarason's Hilbert space formulation of the Julia-Caratheodory theorem [9, Theorem VI-4]:

Theorem 1.5. *Let $\varphi \in \mathcal{S}(n, m)$ and $\zeta \in \partial\mathbb{B}^n$. Then the following are equivalent:*

- (1) *Condition (C).*
- (2) *There exists $\xi \in \partial\mathbb{B}^m$ such that the function*

$$h(z) = \frac{1 - \langle \varphi(z), \xi \rangle}{1 - \langle z, \zeta \rangle}$$

belongs to $\mathcal{H}(\varphi)$.

- (3) *Every $f \in \mathcal{H}(\varphi)$ has a finite K -limit at ζ .*

Proof. First, suppose condition (C) holds. Then there exists a sequence $z_n \rightarrow \zeta$ such that

$$L = \lim \|k_{z_n}^\varphi\|_\varphi^2$$

and by passing to a subsequence we may assume that $\varphi(z_n) \rightarrow \xi$ for some ξ (necessarily $|\xi| = 1$). By weak compactness of the closed unit ball in $\mathcal{H}(\varphi)$ (passing to a further subsequence if necessary) we have $k_{z_n}^\varphi \rightarrow h$ weakly for some $h \in \mathcal{H}(\varphi)$. Thus for all $z \in \mathbb{B}^n$,

$$\begin{aligned} h(z) &= \langle h, k_z^\varphi \rangle_\varphi = \lim_{n \rightarrow \infty} \langle k_{z_n}^\varphi, k_z^\varphi \rangle_\varphi \\ &= \lim_{n \rightarrow \infty} \frac{1 - \langle \varphi(z), \varphi(z_n) \rangle}{1 - \langle z, z_n \rangle} \\ &= \frac{1 - \langle \varphi(z), \xi \rangle}{1 - \langle z, \zeta \rangle} \end{aligned}$$

which proves (2).

Now assume (2). By the lemma, φ has K -limit ξ at ζ ; we will write $\alpha = \varphi(\zeta)$ and k_ζ^φ for the function h in (2). To prove (3) it suffices to prove that $k_z^\varphi \rightarrow k_\zeta^\varphi$ weakly as $z \rightarrow \zeta$ within a Koranyi region. By taking inner products with the kernel functions k_w^φ it is clear that $k_z^\varphi \rightarrow k_\zeta^\varphi$ pointwise on \mathbb{B}^n as $z \rightarrow \zeta$ in a Koranyi region. Since the kernel functions k_w^φ span $\mathcal{H}(\varphi)$, it suffices to prove that the norms $\|k_z^\varphi\|_\varphi$ remain bounded as $z \rightarrow \zeta$ in a Koranyi region. For each $z \in \mathbb{B}^n$ we have

$$\langle k_\zeta^\varphi, k_z^\varphi \rangle = \frac{1 - \langle \varphi(z), \varphi(\zeta) \rangle}{1 - \langle z, \zeta \rangle}$$

so by the Cauchy-Schwarz inequality

$$\left| \frac{1 - \langle \varphi(z), \varphi(\zeta) \rangle}{1 - \langle z, \zeta \rangle} \right|^2 \leq \|k_\zeta^\varphi\|_\varphi^2 \|k_z^\varphi\|_\varphi^2 = \|k_\zeta^\varphi\|_\varphi^2 \left(\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \right)$$

The numerator on the left hand side dominates $(1 - |\varphi(z)|)^2$, so

$$\frac{(1 - |\varphi(z)|)^2}{|1 - \langle z, \zeta \rangle|^2} \leq \|k_\zeta^\varphi\|_\varphi^2 \left(\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \right)$$

which implies

$$\|k_z^\varphi\|_\varphi^2 = \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \leq \|k_\zeta^\varphi\|_\varphi^2 \left(\frac{1 + |\varphi(z)|^2}{1 + |z|^2} \right) \left(\frac{|1 - \langle z, \zeta \rangle|}{1 - |z|} \right)^2$$

The right hand side remains bounded as $z \rightarrow \zeta$ in a Koranyi region, which proves (3).

The proof that (3) implies (1) is immediate, since by the principle of uniform boundedness the norms $\|k_z^\varphi\|_\varphi$ stay bounded as $z \rightarrow \zeta$ in a Koranyi region, which implies condition (C). \square

Theorem 1.6. *Suppose $\varphi \in \mathcal{S}(n, m)$ and satisfies condition (C). Then the function*

$$\frac{1 - |\langle \varphi(z), \xi \rangle|^2}{1 - |\langle z, \zeta \rangle|^2}$$

has K -limit L at ζ .

Proof. By pre- and post-composing with unitary rotations, we may assume without loss of generality that (in the nomenclature of previous theorem) $\xi = e_1$ and $\zeta = e_1$. (We are using e_1 to refer to vectors in two different spaces, but this should cause no confusion.)

Starting with the identity

$$1 - \varphi_1(z) = (1 - z_1) \langle k_{e_1}^\varphi, k_z^\varphi \rangle$$

we find

$$|\varphi_1(z)|^2 = 1 - 2\operatorname{Re}[(1 - z_1) \langle k_{e_1}^\varphi, k_z^\varphi \rangle] + |1 - z_1|^2 |\langle k_{e_1}^\varphi, k_z^\varphi \rangle|^2$$

From what has already been proved, the last term is $o(1 - |z_1|^2)$ as $z \rightarrow e_1$ within a Koranyi region. Thus

$$K\text{-}\lim_{z \rightarrow e_1} \frac{1 - |\varphi_1(z)|^2}{1 - |z_1|^2} = K\text{-}\lim_{z \rightarrow e_1} \frac{2\operatorname{Re}[(1 - z_1) \langle k_{e_1}^\varphi, k_z^\varphi \rangle]}{1 - |z_1|^2}$$

As $z \rightarrow e_1$ in a Koranyi region, the real part of

$$\frac{1 - z_1}{1 - |z_1|^2}$$

tends to $1/2$ and its imaginary part remains bounded. The real part of $\langle k_{e_1}^\varphi, k_z^\varphi \rangle$ tends to $\|k_{e_1}^\varphi\|^2$ and its imaginary part tends to 0. Thus

$$K\text{-}\lim_{z \rightarrow e_1} \frac{2\operatorname{Re}[(1 - z_1) \langle k_{e_1}^\varphi, k_z^\varphi \rangle]}{1 - |z_1|^2} = \|k_{e_1}^\varphi\|^2 = L$$

which completes the proof. \square

Combining statements (2) and (3) of Theorem 1.5 we obtain our first strengthened conclusion, namely that the function h has finite K -limit at ζ . For general φ this will exist only as a restricted K -limit. The same is true for the expression in Theorem 1.6. These facts will allow us to strengthen the convergence results for directional derivatives of the component of φ in the ζ direction.

In the disk, Theorem 1.6 says that $\|k_z^\varphi\|_\varphi \rightarrow \|k_\zeta^\varphi\|_\varphi$ as $z \rightarrow \zeta$ nontangentially; together with the weak convergence of k_z^φ to k_ζ^φ this shows

that in fact $k_z^\varphi \rightarrow k_\zeta^\varphi$ in norm. In the ball we would like to establish $\|k_z^\varphi\|_\varphi \rightarrow \|k_\zeta^\varphi\|_\varphi$ or equivalently

$$\frac{1 - |\varphi(z)|^2}{1 - |z|^2} \rightarrow L$$

in as general a sense as possible. For generic self-maps φ , this limit exists restrictedly but not as a K -limit in general. Unlike the previous results, however, this cannot be improved for Schur-Agler mappings; in fact for the Schur-Agler mapping $\varphi(z) = z_1$ the above expression does not have a K -limit at e_1 . Thus in the ball we only have $\|k_z^\varphi\|_\varphi \rightarrow \|k_\zeta^\varphi\|_\varphi$ (and hence $k_z^\varphi \rightarrow k_\zeta^\varphi$ in norm) when $z \rightarrow \zeta$ restrictedly.

The following is Rudin's version of the Caratheodory theorem on the ball:

Theorem 1.7. *Suppose $\varphi = (\varphi_1, \dots, \varphi_m)$ is a holomorphic mapping from \mathbb{B}^n to \mathbb{B}^m satisfying condition (C) at e_1 . Suppose $2 \leq j \leq m$ and $2 \leq k \leq n$. The following functions are then bounded in every Koranyi region $D_\alpha(e_1)$:*

- (i) $(1 - \varphi_1(z))/(1 - z_1)$
- (ii) $(D_1\varphi_1)(z)$
- (iii) $\varphi_j(z)/(1 - z_1)^{1/2}$
- (iv) $(1 - z_1)^{1/2}(D_1\varphi_j)(z)$
- (v) $(D_k\varphi_1)(z)/(1 - z_1)^{1/2}$
- (vi) $(D_k\varphi_j)(z)$

Moreover, the functions (i), (ii) have restricted K -limit L at e_1 , and the functions (iii), (iv), (v) have restricted K -limit 0 at e_1 .

We next show that for $\varphi \in \mathcal{S}(n, m)$, the restricted K -limits in (i), (ii) and (v) can be improved to K -limits. Note that these are precisely the expressions that involve only the e_1 component of φ . This is to be expected, since the improvement derives from the fact that the kernel k_ζ^φ has a K -limit at ζ , and this kernel depends only on the component of φ in the ζ (that is, the complex normal) direction. Indeed, the limits of (iii) and (iv) cannot be improved to K -limits, since the counterexamples given in [8] are in fact Schur-Agler mappings; this will be shown after proving the next theorem. Before beginning we recall Lemma 8.5.5 of [8] which will be used in the proof.

Lemma 1.8. *Suppose $1 < \alpha < \beta$, $\delta = \frac{1}{3}(1/\alpha - 1/\beta)$, and $z = (z_1, z') \in D_\alpha$.*

- (i) *If $|\lambda| \leq \delta|1 - z_1|$ then $(z_1 + \lambda, z') \in D_\beta$.*
- (ii) *If $|w| \leq \delta|1 - z_1|^{1/2}$ then $(z_1, z' + w') \in D_\beta$.*

Theorem 1.9. *Suppose that $\varphi \in \mathcal{S}(n, m)$ and satisfies condition (C). Then in (i), (ii) and (v) of Theorem 1.7, restricted K -limit can be improved to K -limit.*

Proof. Since we are assuming condition (C), we know from statement (2) of Theorem 1.5 that the function

$$k_{e_1}^\varphi(z) = \frac{1 - \varphi_1(z)}{1 - z_1}$$

belongs to $\mathcal{H}(\varphi)$ and hence by statement (3) has a K -limit at e_1 ; this limit must of course equal L .

For (ii), suppose $1 < \alpha < \beta$, choose δ as in the lemma, let $z \in D_\alpha$ and put

$$r = r(z) = \delta|1 - z_1|$$

As in [8], express $D_1\varphi_1$ using the Cauchy formula; after some manipulation we obtain

$$(3) \quad (D_1\varphi_1)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \varphi_1(z_1 + re^{i\theta}, z')}{1 - (z_1 + re^{i\theta})} \cdot \left\{ 1 - \frac{1 - z_1}{re^{i\theta}} \right\} d\theta$$

We must show that the above expression tends to L along any sequence converging to e_1 within D_α ; in fact it suffices to show that given any such sequence, $D_1\varphi_1$ converges to L along some subsequence. In particular we may assume that we have chosen a sequence (z_n) such that

$$\lim_{n \rightarrow \infty} \frac{1 - z_{n,1}}{r(z_n)e^{i\theta}} = \frac{1}{\delta e^{i\theta}} \lim_{n \rightarrow \infty} \frac{1 - z_{n,1}}{|1 - z_{n,1}|}$$

exists, and is equal to some complex number λ . Then as $z_n \rightarrow e_1$ in D_α , we have

$$z_n + r(z_n)e^{i\theta}e_1 \rightarrow e_1$$

in D_β , so the integrand in (3) converges to

$$L \cdot \left(1 - \frac{\lambda}{\delta e^{i\theta}} \right)$$

for every $\theta \in [0, 2\pi]$. Since this integrates to L , and the integrands are uniformly bounded, we conclude $D_1\varphi_1(z_n) \rightarrow L$ by the dominated convergence theorem.

The K -limit of (v) is established similarly: we let α, β, δ be as before, and for $z \in D_\alpha(e_1)$ we define

$$\rho = \rho(z) = \delta|1 - z_1|^{1/2}$$

Then by the lemma, $(z_1, z' + w') \in D_\beta(e_1)$ for all w' with $|w'| \leq \rho$. Assuming $k = 2$ (without loss of generality), we apply the Cauchy

formula to obtain for every $z \in D_\alpha$

$$\frac{(D_2\varphi_1)(z)}{(1-z_1)^{1/2}} = -\frac{(1-z_1)^{1/2}}{\rho(z)} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-\varphi_1(z_1, z_2 + \rho e^{i\theta}, \dots)}{1-z_1} e^{i\theta} d\theta$$

The factor outside the integral is bounded. As $z \rightarrow e_1$ within D_α , $z + \rho(z)e^{i\theta}e_2 \rightarrow e_1$ within D_β , so the integrand tends to $Le^{i\theta}$ for every θ . Thus

$$\frac{(D_2\varphi_1)(z)}{(1-z_1)^{1/2}} \rightarrow 0$$

by the dominated convergence theorem. \square

Rudin [8] gives counterexamples to show that “restricted K -limit” cannot be improved to “ K -limit” in Theorem 1.7; in the case of (iii) and (iv), the example is a map $\varphi : \mathbb{B}^2 \rightarrow \mathbb{B}^2$ of the form

$$(4) \quad \varphi(z_1, z_2) = (z_1, z_2 g(z_1))$$

for a suitably chosen holomorphic function $g : \mathbb{D} \rightarrow \mathbb{D}$. It is not hard to show that any map of the form (4) belongs to $\mathcal{S}(2, 2)$. To see this, first observe that because $g : \mathbb{D} \rightarrow \mathbb{D}$, the kernel

$$\frac{1 - g(z)\overline{g(w)}}{1 - z\overline{w}}$$

is positive. We may then write

$$(5) \quad \frac{1 - \langle \varphi(z), \varphi(w) \rangle}{1 - \langle z, w \rangle} = \frac{1 - \langle z, w \rangle + z_2 \overline{w_2} - z_2 \overline{w_2} g(z_1) \overline{g(w_1)}}{1 - \langle z, w \rangle}$$

$$(6) \quad = 1 + z_2 \overline{w_2} \frac{1 - g(z_1) \overline{g(w_1)}}{1 - z_1 \overline{w_1}} \cdot \frac{1 - z_1 \overline{w_1}}{1 - \langle z, w \rangle}$$

which is positive.

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